Tutorial 4 (Questions)

Recall the Minimax Theorem. This existence theorem also gives a characterization of the value of a game matrix and optimal strategies for the two players. More precisely, given an $m \times n$ matrix A, we call a number v the value of A, a probability vector $\boldsymbol{p} \in \mathcal{P}^m$ a maximin strategy for the row player, and a probability vector $\boldsymbol{q} \in \mathcal{P}^n$ a minimax strategy for the column player if

(i) $\boldsymbol{p}A\boldsymbol{y}^T \geq v$ for any $\boldsymbol{y} \in \mathcal{P}^n$.

(ii) $\boldsymbol{x}A\boldsymbol{q}^T \leq v$ for any $\boldsymbol{x} \in \mathcal{P}^m$.

(iii) $\boldsymbol{p}A\boldsymbol{q}^T = v$.

We note condition (i) is equivalent to

(i)′ every element of the row vector $\boldsymbol{p}A$ is at least v,

and the condition (ii) is equivalent to

(ii)′ every element of the column vector $A\boldsymbol{q}^T$ is at most v.

Exercise 1. *Let A be an $m \times m$ matrix and B be an $n \times n$ matrix. Let M be the $(m+n) \times (m+n)$ matrix given by*

$$M = \begin{pmatrix} A & O \\ O & B \end{pmatrix}.$$

Let u be the value, $\boldsymbol{p} \in \mathcal{P}^m$ be a maximin strategy for the row player and $\boldsymbol{q} \in \mathcal{P}^m$ be a minimax strategy for the column player of A. Let v be the value, $\boldsymbol{r} \in \mathcal{P}^n$ be a maximin strategy for the row player and $\boldsymbol{s} \in \mathcal{P}^n$ be a minimax strategy for the column player of B.

(i) Suppose $u > 0$ and $v < 0$. Find the value of M and optimal strategies for the two players of the game with game matrix M.

(ii) Suppose $u > 0$ and $v > 0$. Find the value of M in terms of u and v. Find optimal strategies for the row player and the column player of M in terms of $u, v, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$.

Solution. (i) Note that

$$\begin{pmatrix} \boldsymbol{p} & \mathbf{0} \end{pmatrix} \begin{pmatrix} A & O \\ O & B \end{pmatrix} = \begin{pmatrix} \boldsymbol{p}A & \mathbf{0} \end{pmatrix},$$

and

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}^T \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ B\boldsymbol{s}^T \end{pmatrix}.$$

Since $u > 0$ and $\boldsymbol{p}$ is an maximin strategy for the row player of A, we have every element of the $m + n$ dimensional row vector $(\boldsymbol{p}A, \mathbf{0})$ is at least 0. Similarly, since $v < 0$, we have every element of the $m + n$ dimensional column $(\mathbf{0}, \boldsymbol{s}B^T)^T$ is at most 0. Clearly,

$$\begin{pmatrix} \boldsymbol{p} & \mathbf{0} \end{pmatrix} \begin{pmatrix} A & O \\ O & B \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}^T \end{pmatrix} = 0.$$

Hence by the Minimax Theorem, the value of M equals 0, $(\boldsymbol{p}, \mathbf{0})$ is a maximin strategy for the row of player of M and $(\mathbf{0}, \boldsymbol{s})$ is a minimax strategy for the column player of M.

(ii). In the case that $u, v > 0$, we start by assuming that for some $\lambda \in [0, 1]$ (to be determined), $(\lambda\boldsymbol{p}, (1-\lambda)\boldsymbol{r})$ and $(\lambda\boldsymbol{q}, (1-\lambda)\boldsymbol{s})$ are optimal strategies for the row player and the column player of M respectively.

Consider

$$\begin{pmatrix} \lambda\boldsymbol{p} & (1-\lambda)\boldsymbol{r} \end{pmatrix} \begin{pmatrix} A & O \\ O & B \end{pmatrix} = \begin{pmatrix} \lambda\boldsymbol{p}A & (1-\lambda)\boldsymbol{r}B \end{pmatrix}.$$

By the definition of $\boldsymbol{p}$ and $\boldsymbol{r}$, we have each of the first m coordinates of $(\lambda \boldsymbol{p}A, (1-\lambda)\boldsymbol{r}B)$ is at least λu, and each of the last n coordinates of $(\lambda \boldsymbol{p}A, (1-\lambda)\boldsymbol{r}B)$ is at least $(1-\lambda)v$. Since $u, v > 0$, by letting $\lambda u = (1-\lambda)v$, we have $\lambda = \frac{v}{u+v}$ and $\lambda u = \frac{uv}{u+v}$. Then we have each element of the vector

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix} \begin{pmatrix} \frac{v}{u+v}\boldsymbol{q}^T \\ \frac{u}{u+v}\boldsymbol{s}^T \end{pmatrix} = \begin{pmatrix} \frac{v}{u+v}A\boldsymbol{q}^T \\ \frac{u}{u+v}B\boldsymbol{s}^T \end{pmatrix}$$

is at most $\frac{uv}{u+v}$. More over,

$$\begin{pmatrix} \frac{v}{u+v}\boldsymbol{p} & \frac{u}{u+v}\boldsymbol{r} \end{pmatrix} \begin{pmatrix} A & O \\ O & B \end{pmatrix} \begin{pmatrix} \frac{v}{u+v}\boldsymbol{q}^T \\ \frac{u}{u+v}\boldsymbol{s}^T \end{pmatrix} = \frac{uv}{u+v}.$$

Hence by the Minimax Theorem, we have the value of M is $\frac{uv}{u+v}$, $(\frac{v}{u+v}\boldsymbol{p}, \frac{u}{u+v}\boldsymbol{r})$ is an optimal strategy for the row player and $(\frac{v}{u+v}\boldsymbol{q}, \frac{u}{u+v}\boldsymbol{s})$ is an optimal s-trategy for the column player.

Exercise 2. *Let*

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix},$$

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

(i) Suppose $\lambda_1 \leq 0$ *and* $\lambda_n > 0$. *Find the value of* A.

(ii) Suppose $\lambda_1 > 0$. *Solve the two-person zero-sum game with game matrix* A.

Solution. (i) Let $k \geq 1$ be the smallest integer such that $\lambda_k \leq 0$ and $\lambda_{k+1} > 0$. Set $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{P}^n$ by

$$\boldsymbol{p} = (0, \cdots, 0, p_{k+1}, \cdots, p_n), \ \boldsymbol{q} = (q_1, \cdots, q_k, 0, \cdots, 0).$$

Then by the choice of k, we have

(a) $\boldsymbol{p}A = (0, \cdots, 0, p_{k+1}\lambda_{k+1}, \cdots, p_n\lambda_n)$ has all elements ≥ 0.

(b) $A\boldsymbol{q}^T = (q_1\lambda_1, \cdots, q_k\lambda_k, 0, \cdots, 0)^T$ has all elements ≤ 0.

(c) $\boldsymbol{p}A\boldsymbol{q}^T = 0$.

Hence by the Minimax Theorem, the value of A equals 0.

(ii) Let v denote the value of A. Assume $\boldsymbol{p} = (p_1, \cdots, p_n)$ is an optimal strategy for the row player. Assume $p_k > 0$ for $1 \leq k \leq n$. Then by the principle of indifference, we have

$$\begin{pmatrix} p_1 & \cdots & p_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \begin{pmatrix} v & \cdots & v \end{pmatrix},$$

which implies (since $\lambda_k > 0$ for all k)

$$v = \frac{1}{\frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n}},$$

and

$$p_k = \frac{\frac{1}{\lambda_k}}{\frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n}}, \text{ for } k = 1, \cdots, n.$$

Assume $\boldsymbol{q} = (q_1, \cdots, q_n)$ ($q_k > 0$ for all k) is an optimal strategy for the column player, it is easy to see $\boldsymbol{p} = \boldsymbol{q}$. Clearly, for the above v, $\boldsymbol{p}$ and $\boldsymbol{q}$, the conclusion of the Maximin Theorem holds. Hence $v, \boldsymbol{p}, \boldsymbol{q}$ are desired.

Exercise 3. *Player I and Player II choose integers i and j respectively from the set $\{1, \cdots 7\}$. Player I winns 1 dollar if $|i - j| = 1$, otherwise there is no payoff. Find the game matrix and solve the game.*

Solution. The game matrix is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

By deleting dominated rows and columns, we obtain the reduced matrix

$$A' = \begin{array}{c} \\ 2 \\ 3 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 6 & 7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

By the principle of indifference, it is easy to see the value of A is $\frac{1}{4}$, and $(0, \frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4}, 0)$ is the only optimal strategy for the row player, $(\frac{1}{4}, \frac{1}{4}, 0, 0, 0, \frac{1}{4}, \frac{1}{4})$ is the only optimal strategy for the column player.